International Scientific Physics Olympiad

T2. Muon anomalous magnetic moment

A1 For a muon in the beam find the difference between speed of light and muons' speed $\Delta v = c - v$ (provide formula and compute numerical value).

Energy of a relativistic particle can be expressed in terms of its momentum

$$E = \sqrt{p^2c^2 + m_\mu^2 c^4},$$

and velocity equals to

$$v = \frac{c^2 p}{E} = \frac{cp}{\sqrt{p^2 + m_\mu^2 c^2}}.$$

For the speed difference we have

$$\Delta v = c - v = c\left(1 - \frac{p}{\sqrt{p^2 + m_\mu^2 c^2}}\right) \approx \frac{1}{2}c\left(\frac{m_\mu c}{p}\right)^2 \approx 1.75 \cdot 10^5 \text{ m/s}.$$

For the given numerical values we can use the approximate formula, it gives deviation from the exact value only in the fourth figure.

A2 For a muon in the beam find the kinetic energy E_k, that is the difference between the full energy and the rest energy (provide formula and compute numerical value in Gev).

We use expression for the energy from the previous question. Note that for the given numerical values one can use approximate expression for the ultrarelativistic particle $E \approx pc$.

$$E_k = \sqrt{p^2c^2 + m_\mu^2 c^4} - m_\mu c^2 \approx 2.99 \text{ GeV}$$

A3 Find radius R of muon's orbit and period T of its motion. (Provide formulas and compute numerical values.)

From the relativistic equation of motion is

$$\frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B} = \frac{q}{\gamma m_\mu}\vec{p} \times \vec{B}$$

it follows that the absolute values of velocity and momentum are constant . Therefore momentum rotates with

$$\frac{d\vec{p}}{dt} = \vec{\omega}_c \times \vec{p}, \quad \vec{\omega}_c = -\frac{q\vec{B}}{\gamma m}.$$

Here ω_c is muon angular velocity. The period is

$$T = \frac{2\pi}{\omega_c} = \frac{2\pi m\gamma}{qB} = 1.49 \cdot 10^{-7} \text{ s}.$$

The orbit radius is

$$R = \frac{vT}{2\pi} = \frac{\gamma m}{qB}v = \frac{\gamma m}{qB}\frac{pc^2}{E} = \frac{p}{qB} = 7.12 \text{ m}.$$

A4 Due to instability a muon can orbit the circle only finite time. Find number of revolutions N the muon makes before its decay.

Due to relativistic time dilation the muon lifetime in laboratory reference frame is $t = \gamma\tau$, therefore the number of revolutions is

$$N = \frac{t}{T} = \gamma\tau\frac{qB}{2\pi m_\mu \gamma} = \frac{qB\tau}{2\pi m_\mu} = 431.$$

A5 Obtain a formula for the spin time derivative, $d\vec{s}/dt$, express the answer in terms of $\vec{B}$, $\vec{s}$, g_μ and fundamental constants.

Torque acting on a magnetic dipole in magnetic field is

$$\vec{M} = \vec{\mu} \times \vec{B},$$

thus equation of motion are

$$\frac{d\vec{s}}{dt} = \vec{M} = \vec{\mu} \times \vec{B} = g_\mu \frac{q}{2m_\mu} \vec{s} \times \vec{B} = -g_\mu \frac{q\vec{B}}{2m_\mu} \times \vec{s}.$$

A6 Find the z component of angular velocity ω_z of muon's spin. Express the answer in terms of B, g_μ and fundamental constants.

From previous question it follows that the angular velocity is

$$\vec{\omega} = -g_\mu \frac{q\vec{B}}{2m_\mu}.$$

Therefore z component of angular velocity is

$$\omega_z = -g_\mu \frac{qB}{2m_\mu}.$$

B1 Find the x and y components of velocity $\vec{v}_C$ of reference frame C with respect to the reference frame A. Express the answer in terms of v_x, v'_y.

Recall the formula for the adding velocity from the task. In our case $v'_x = 0$ (frame C velocity component along x axis), therefore denominator is $1 + v'xv_x/c^2 = 1$.

$$v_{Cx} = v_x, \quad v_{Cy} = v'_y \sqrt{1 - v_x^2/c^2} = v'_y/\gamma.$$

B2 Find the x'' and y'' components of velocity $\vec{v}''_A$ of the reference frame A with respect to the reference frame C. Take into account that v''_y is small.

Note that if frame B moves relatively to A with velocity $\vec{v}$, then fram A moves relatively frame B with velocity $-\vec{v}$. Therefore frame B velocity with respect to frame C is $-v''_y$, and frame A velocity with respect to frame B is $-v_x$. The γ coefficient is calculated with velocity v'_y:

$$\gamma = \frac{1}{\sqrt{1 - v_y'^2/c^2}} \approx 1,$$

since we have to calculate only terms livear in v'_y. The velocities of frames A and B are orthogonal and the denominator in velocity addiction formula is equal to 1. The velocity of frame A with respect to frame C is

$$v''_{Ax} = -v_x, \quad v''_{Ay} = -v'_y.$$

B3 It turns out that $\vec{v}''_A \neq -\vec{v}_C$. It can be attributed to the fact that axes of the reference frame C are rotated with respect to the axes of frame A. Therefore all vectors in frame C are also rotated with respect to the frame A. Calculate the angle of rotation $\Delta\theta$ defined as an angle between vectors $-\vec{v}''_A$ and $\vec{v}_C$. The angle is positive if the rotation from first vector to the second vector viewed from the z axis is counterclockwise. (Axes x, y, z form right-handed coordinate system.) Express your answer in terms of v_x, v'_y, $\gamma = 1/\sqrt{1 - v_x^2/c^2}$, c. **0.6**

The angle between two vector can be calculated from their vector product (length of vector product is $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\alpha$, where α is the angle between the vector). Let $\vec{e}_x$ be an unit vector in x direction, $\vec{e}_y$ – in y direction, $\vec{e}_z$ – in z direction. The vector product is

$$-\vec{v}''_A \times \vec{v}_C = (v_x\vec{e}_x + v'_y\vec{e}_y) \times (v_x\vec{e}_x + (v'_y/\gamma)\vec{e}_y) = v_x v'_y \left(\frac{1}{\gamma} - 1\right) \vec{e}_z.$$

Both vectors are parallel to xy plane, therefore the vector product is collinear with z axis, positive z component corresponds to the positive angle of rotation. Up to the terms linear in v'_y we can approximate $|\vec{v}''_A| = |\vec{v}_C| = v_x$, thus angle of rotation is

$$\Delta\theta = \frac{1}{v_x^2} v_x v'_y \left(\frac{1}{\gamma} - 1\right) = -\frac{v'_y}{v_x}\left(1 - \frac{1}{\gamma}\right).$$

B4 Let a particle move with velocity $\vec{v}$ along x axis with respect to the frame A. Particle's acceleration is directed along the y axis, its projection is a_y. Frame B is the comoving frame for the particle at the considered moment (i.e. the velocity of the particle in it is zero) and moves with the speed v along x axis. After time interval dt (measured in laboratory reference frame) particle velocity is $\vec{v} + d\vec{v}$, corresponding comoving reference frame is C. From the previous question it follows that the axes of comoving frame rotate. Using results of the previous quaestion, find z component ω_T (angular velocity of Thomas precession) of the comoving frame angular velocity (in this case rotation is around z axis). Express the answer in terms of v_x, a_y, γ.

After time dt in laboratory frame of reference the particle velocity changes by $v_y = a_y dt$, in the frame B, which is comoving for the particle in the initial moment of time, corresponding velocity is $v'_y = \gamma v_y$. This result can be obtained from B1, or directly from velocity addiction formula (laboratory frame of reference moves relatively to frame B with velocity $-v_x$, particle velocity with respect to the laboratory frame of reference is $\vec{v} = (v_x, v_y)$):

$$v'_y = \frac{v_y\sqrt{1 - v_x^2/c^2}}{1 - v_x^2/c^2} = \frac{v_y}{\sqrt{1 - v_x^2/c^2}} = \gamma v_y.$$

Thus the rotation angle

$$d\theta = -\frac{\gamma a_y dt}{v_x}\left(1 - \frac{1}{\gamma}\right),$$

and angular velocity

$$\omega_T = \frac{d\theta}{dt} = -\frac{a_y}{v_x}(\gamma - 1).$$

Note that according our sign conventions positive angle of rotation corresponds to positive z component of the angular velocity.

B5 Let the muon move in homogeneous magnetic field B directed along z axis. Muon velocity is perpendicular to the magnetic field. Calculate the z component of Thomes precession angular velocity. Express the answer in terms of B, q, m_μ, γ.

From the equation of motion in magnetic field

$$\mu\frac{d\vec{p}}{dt} = m_\mu \frac{d(\gamma\vec{v})}{dt} = q\vec{v} \times \vec{B}$$

we calculate the acceleration component (note that γ is constant)

$$a_y = -\frac{q v_x B}{\gamma m_\mu},$$

thus angular velocity is

$$\omega_T = \frac{qB}{\gamma m_\mu}(\gamma - 1).$$

C1 Find the expression for z component of muon spin precession ω_s angular velocity with respect to laboratory reference frame. Express the answer in terms of B, g_μ, q, m_μ, γ.

The full angular velocity of precession is sum of the angular velocity of precession in magnetic field and Thomas precession angular velocity:

$$\omega_s = \omega_z + \omega_T = -g_\mu \frac{qB}{2m_\mu} + \frac{qB}{\gamma m_\mu}(\gamma - 1)$$

C2 In practice it is more convenient to consider spin rotation with respect to the direction of muon momentum. Find the angular velocity ω_a of muon spin rotation with respect to the direction of its momentum (that is also equal to the time derivative of the angle between momentum and spin). Express the answer in terms of B, a_μ, q, m_μ, γ.

The muon momentum rotates with anguler velocity which is equal to the angular velocity of the muon in magnetic field ω_c, which was calculated in A3, its z component is

$$\omega_z = -\frac{qB}{\gamma m_\mu}$$

Therefore the angular velocity of the spin with respect to the momentum is equal to the difference between the angular velocities of the spin and momentum (all rotations are arond z axis):

$$\omega_a = \omega_s - \omega_c = -g_\mu \frac{qB}{2m_\mu} + \frac{qB}{\gamma m_\mu}(\gamma - 1) + \frac{qB}{\gamma m_\mu} = -g_\mu \frac{qB}{2m_\mu} + \frac{qB}{m_\mu}.$$

By the relation $g_\mu = 2(1 + a_\mu)$, we obtain

$$\omega_a = -a_\mu \frac{qB}{m_\mu}.$$

C3 Experimentally measured frequency of muon spin precession with respect to the momentum direction is $f_a = 229081$ Hz. Find the expression for muon anomalous magnetic moment a_μ. Express the answer in terms of f_a, B, q, m_μ, γ. Compute the numerical value of a_μ.

Frequency in terms of angular velocity is given by $f_a = \omega_a/2\pi$, therefore

$$a_\mu = \frac{m_\mu \omega_a}{qB} = \frac{2\pi f_a m_\mu}{qB} = 0.00117.$$

The direction of the spin precession is not given, therefore we cannot determine the sign of a_μ, it turns out that $a_\mu > 0$.