T3. Frequency measurements with ring resonators

This task deals with methods for high-precision frequency measurements. During the solution, you will learn two measurement methods: signal modulation (part C) and frequency comb (part D).

Why do you need more precision?

- The accuracy of physical measurements is important when we determine the values of fundamental constants. Many times in the history of physics, small discrepancies between experimental results and theory have required revision of the entire theoretical model and led to major discoveries in fundamental physics.
- High accuracy of frequency measurement is necessary for normal operation of satellite communication systems, navigation, ground-based telecommunication systems, etc.

In the framework of classical physics, any measurement is a comparison with a standard. As a frequency standard we will use the natural frequencies of a **ring optical resonator,** because today the best resonators are ring resonators. And the higher the goodness, the more accurate is the value of the reference natural frequency. In Part B, you will learn more about ring resonators with the example of **a ring fiber optical resonator**. In Part A, you will derive the theory of wave propagation in the optical fiber from which the resonator described in Part B is made. At the very end of the problem there are reference materials: Maxwell's equations and differential operators. If you know all the physical laws that are included in the Olympiad program, you will **not need reference materials to solve the problem**.

Equations of EM wave propagation The electromagnetic field at each point of space is described by four vectors: $\vec{E}(\vec{r},t)$, $\vec{H}(\vec{r},t)$, $\vec{D}(\vec{r},t)$, $\vec{B}(\vec{r},t)$ — electric field strength, magnetic field strength, electric induction and magnetic induction. These vectors are related to the $\vec{P}$ and magnetisation $\vec{M}$ of the medium by realations:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P},$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}),$$

where ε_0 is the electric constant and μ_0 is the magnetic constant.

Use following approximations when solving the problem:

- All media are non-magnetic and $\vec{M} = 0$.
- There are no free charges.
- The response of the medium is instantaneous and local, that is, $\vec{P}(\vec{r},t)$ depends only on $\vec{E}(\vec{r},t)$.

EM wave theoey. By applying the rot operator to Maxwell's equations, Faraday's induction law in differential form and the theorem on the circulation of magnetic field strength in differential form (see References), we obtain the basic equations of electromagnetic wave propagation:

$$\Delta \vec{E} = \mu_0 \frac{\partial^2 \vec{D}}{\partial t^2}.$$

$$\Delta \vec{H} = \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}.$$

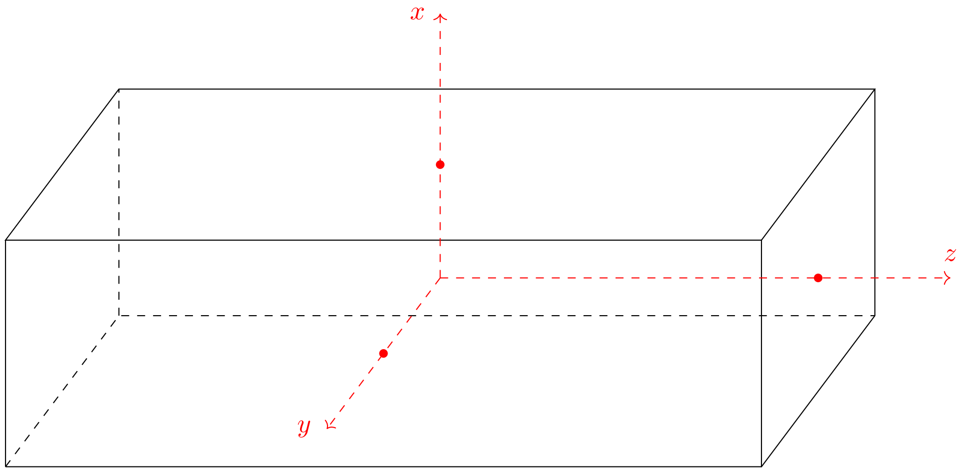
Part A. Field profile in fiber optics (2.5 points)

In parts $\mathrm{A, B, C}$, we will assume that the polarization depends linearly on the electric field strength: $\vec{P} = \varepsilon_0(\varepsilon(\omega) - 1)\vec{E}$, where $\varepsilon(\omega)$ is the dielectric permittivity of the medium, which usually depends on the frequency of the electromagnetic wave in the medium. Dielectric permittivity of vacuum is equal to 1.

So, the equations of propagation of electromagnetic waves can be written in this form:

$$\Delta \vec{E} = \mu_0 \varepsilon \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}.$$

$$\Delta \vec{H} = \mu_0 \varepsilon \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2}.$$



A plane monochromatic electromagnetic wave of frequency ω propagates along a dielectric waveguide along the z-axis (fig. 1), which is a rectangular parallelepiped of material with dielectric constant $\varepsilon(\omega)$ with sides a, L_1 и L_2. $L_1, L_2 \to \infty$ and are much more than any characteristic lengths in this problem.

Let's introduce the Cartesian coordinate system $Oxyz$. O ccoincides with the center of the parallelepiped. The edges of length L_1 are parallel to z-axis, length a – to x-axis, length L_2 – to y-axis. $(\hat{x}, \hat{y}, \hat{z})$ is a right triple.

Consider a wave polarized parallel to the y-axis: $\vec{E}||\hat{y}$. We will look for the solution for the field $\vec{E}$ in the waveguide in the following form:

$$\vec{E}(\vec{r}, t) = E_0 \,\mathrm{Re}\left(A(x, y)\exp(i\omega t - i\beta z)\right)\hat{y},$$

where E_0 has the dimensionality of the electric field, and $A(x, y)$ is a dimensionless complex quantity that contains information about the ratio of amplitudes and phase differences of the field at different points of the waveguide. Without restriction of generality $A(0, 0) = 1$.

Let us also consider that $L_2 \gg a$, so it will be assumed that $\partial A/\partial y \ll \partial A/\partial x$, and hence A is only dependent on x.

The solution for $A(x)$ can be sought in the form:

$$A(x) = \begin{cases} \cos(k_{in}x) \quad \text{при} \quad |x| < a/2 \\ C_1 \exp(k_{out}x) + B\exp(-k_{out}x) \quad \text{при} \quad x > a/2 \\ B\exp(k_{out}x) + C_2\exp(-k_{out}x) \quad \text{при} \quad x < -a/2 \end{cases}$$

where $k_{in} > 0, k_{out} > 0$.

A1	What are the coefficients C_1 and C_2?	**0.3**

A2	Express k_{in} and k_{out} in terms of $\varepsilon, \omega, \beta$ and speed of light in vacuum $c = 1/\sqrt{\varepsilon_0\mu_0}$, using the wave equation.	**0.5**

To find B, we need to write down the boundary conditions for the boundary $x = a/2$ (for the boundary $x = -a/2$ we get the same equations because $A(x)$ is an even function).

A3	Prove that the tangential components of the electric field strength are equal on both sides of the interface (as in electrostatics).	**0.3**

A4	Write the boundary conditions for the tangential component of the electric field. Express B in terms of a, k_{in}, k_{out}.	**0.3**

Now we need to derive an equation for finding $\beta(\omega)$. For this purpose we can write another boundary condition – for $\vec{H}$.

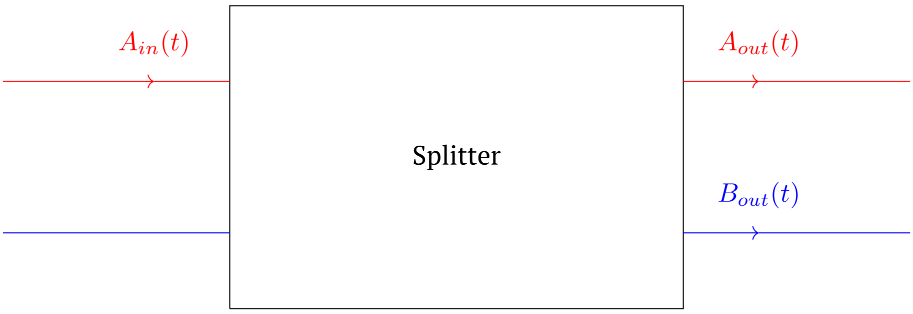
A5	Prove that the tangential components of the magnetic field strength are equal on both sides of the interface (as in magnetostatics).	**0.3**

A6	Prove that $$H_z = E_0 \,\mathrm{Re}\left(\frac{i}{\mu_0\omega}\frac{\partial A}{\partial x}\exp(i(\omega t - \beta z))\right).$$	**0.3**

A7	Write the boundary conditions for the tangential component of the magnetic field $\vec{H}$. The answer can be expressed in terms of k_{in}, k_{out}, a.	**0.3**

A8	Substitute the values k_{in}, k_{out}, obtained in A2 into the equation from A7. Obtain an equation from which βcan be determined (this equation is solved numerically only). The equation can include $\beta, \omega, \varepsilon, c, a$.	**0.2**

Part B. Fiber resonator transmittance (3.0 points)

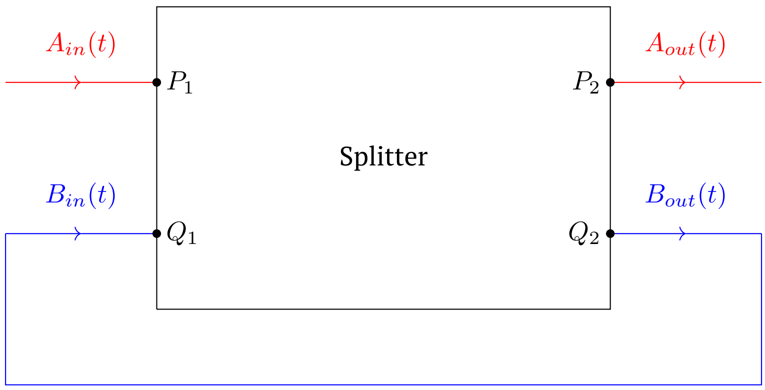


The radiation from the external laser enters a quadrupole beam splitter, which is an optical system with two pairs of coupled channels: (P_1, P_2) and (Q_1, Q_2).

A monochromatic wave with complex amplitude $A_{in}(t)$ entering the splitter is split into two outgoing waves of smaller amplitude propagating in the same direction. The first of them propagates along the same pair channel as the incoming wave, and its complex amplitude is $A_{out}(t) = -r_s A_{in}(t)$. The second wave propagates through the channel of the other pair and its amplitude is $B_{out}(t) = it_s A_{in}(t)$. **Waves incident on Q_1 are divided simillary for those incident on P_1.** Consider that r_s and t_s are real positive numbers and $t_s \ll 1$.

B1	Assuming that there is no energy loss in the divider, find the relationship between r_s and t_s.	**0.2**

A fiber resonator can be assembled with the divider. In order to do so, it is necessary to connect the channels Q_1 and Q_2 with a loop of single-mode fiber (the equation for $\beta(\omega)$ has the only solution for the frequencies considered in this part) of length L. The time for light of frequency ω to travel through the loop is $\tau(\omega)$. Also note that due to material absorption, the amplitude of the wave in the loop decreases by a factor $1/\kappa$ for each round in the loop $(1-\kappa \ll 1)$. For all tasks in parts B and C **neglect backscattering**.



Let a monochromatic wave $A_{in}(t) = A_0 e^{i\omega t}$ from an external laser be incident on the input of channel P_1.

B2	Express the field amplitude at the input to Q_1 of the splitter $B_{in}(t)$ in terms of κ and the field amplitude at the output of Q_2 at time $(t-\tau(\omega))$ — $B_{out}(t-\tau(\omega))$.	**0.3**

Assume the system has come to the steady-state regime, i.e. the absolute values of complex field amplitudes at any point are constant, and the phase difference of the field at two points does not change with time. Then for any amplitude E_i (instead of E_i we can substitute $A_{in}, A_{out}, B_{in}, B_{out}$) the expressions are valid:

$$E_i(t - t') = e^{-i\omega t'} E_i(t).$$

B3	Using the stationarity conditions, express $B_{in}(t)$ in terms of $B_{out}(t), \kappa, \omega, \tau$.	**0.3**

B4	Express $B_{in}(t)$ in terms of $A_0, r_s, t_s, \kappa, \omega, \tau$ and t, using the result of the previous task.	**0.5**

B5	What is the power N_2, leaving the channel P_2? Express the answer in terms of $\omega\tau(\omega), \kappa, r_s$ and the power N_1 in channel P_1.	**0.5**

B6 Sketch a qualitative plot of $N_2/N_1(\omega\tau)$ for fiber resonator with the following parameters: **0.6**

- $\kappa = 1 - 5 \cdot 10^{-3}$;
- $t_s = 0.1$.

At what values of $\omega\tau$ is the ratio N_2/N_1 minimal?

B7 Find the sharpness Q of the absorption peak with number $n = 100$. **0.6**
Sharpness is the ratio of the peak frequency to the width of the region of frequencies for which the transmission dip is not less than half of the maximum dip of a particular peak.

Today it is possible to create a fiber resonator based on a beam splitter with peak sharpness of about $5 \cdot 10^7$ near the frequency corresponding to the wavelength of 1550 nm. And if we use a ring microresonator made of fused quartz, we can obtain a sharpness of 10^9. Resonators with such small losses open us a wide range of possibilities for accurate frequency measurements.

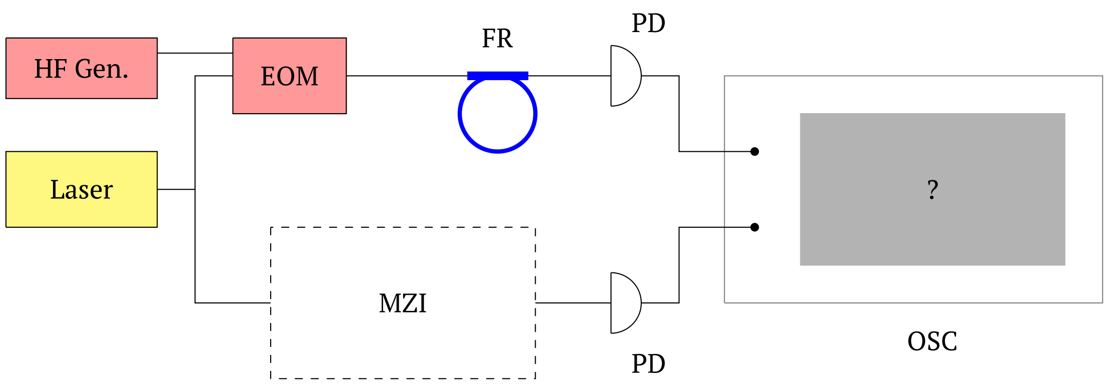
That is, with a quartz microresonator we can obtain a spectral line with an average wavelength of 1550 nm with a width of only 0.000002 nm!

Part C. Calibration of the Mach-Zehnder Interferometer (2.0 points)

Consider the setup for calibrating the interferometer shown in Fig.1 Now the laser frequency varies smoothly with time (the derivative $\omega\tau$ is much smaller than the inverse time for establishing the stationary mode in the resonator), and its power is constant.

$$\omega(t) = \omega_0 + \alpha t$$

The radiation flux from the laser is split into two arms by another beam splitter similar to the splitter in part B. The first arm of the optical circuit contains a Mach-Zehnder calibrated interferometer and a photodetector that measures the power coming out of it, and the second arm contains a resonator and a similar photodetector. The signals from the photodetectors are fed to an oscilloscope. The signal of the photodetector is proportional to the radiation power. **The oscilloscope shows the time dependence of the photodetector voltages (not XY mode!).**



Laser parameters:

- $\omega_0 = 1.26 \cdot 10^{15}$ с$^{-1}$
- $\alpha = 5 \cdot 10^9$ с$^{-2}$

Resonator parameters:

- Sharpness of absorption peaks near ω_0 is $5 \cdot 10^7$
- Consider that $\tau(\omega)$ independent of frequency and equal to $\tau_0 = 3 \cdot 10^{-11}$ с
- $\kappa = 1 - 1 \cdot 10^{-3}$
- $t_s = 0.1$

Parameters of the interferometer to be calibrated:

- Its **power** throughput capacity is approx

$$F_{MZI}(\omega) = \cos^2(\omega/2\omega_{MZI})$$

- $\omega_{MZI} \approx 1250MHz$. The exact value will be determined at this setup.

C1	Sketch a qualitative plot of the oscilloscope readings if it is known for sure that the frequency of the tunable laser reaches exactly one of the frequencies given in B6 (these are the frequencies where the FR transmittance is minimal).	**0.5**

Now an amplitude electro-optic modulator is connected to the second arm. Its **amplitude** bandwidth is independent of the radiation wavelength and is equal to:

$$f_{EOM}(t) = \beta + (1 - \beta)\cos(\Omega t)$$

$$\Omega \ll \omega_0$$

$$\beta < 1$$

C2	Draw what the oscilloscope will show when $\Omega \approx 220MHz$. Note that $\alpha \ll \Omega\omega_0$.	**1.0**

C3	Estimate with what relative accuracy can the period ω_{MZI} be measured on this setup if the maximum signal frequency of the high-frequency oscillator is $\Omega_{max} = 1250MHz$?	**0.5**

Part D. Cubic nonlinearity and solitons (2.5 points)

For precision measurements and high-frequency communication, it is necessary to obtain very short light pulses that do not change their profile with time (such pulses are called solitons). This is not possible in linear optics because the dielectric constant and the dispersion constant depend on frequency, and the pulses do "blur"due to dispersion.

Fortunately, the approximations of linear optics do not work for all materials, and the resulting nonlinearities can compensate for the undesirable effects associated with dispersion.

In Part D, we will assume that the projection $P_i(\vec{r}, t)$ of the polarisation vector on an arbitrary axis i depends on the projection of the electric field strength $E_i(\vec{r}, t)$ at the same point at that instant of time as follows:

$$P_i(\vec{r}, t) = \varepsilon_0((\varepsilon(\omega) - 1)E_i(\vec{r}, t) + \chi E_i^3(\vec{r}, t))$$

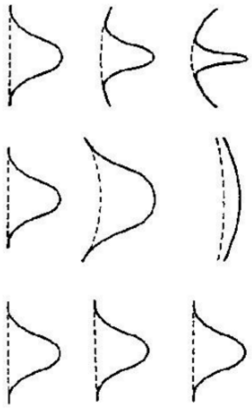
The pulse is a sum of electromagnetic waves with frequencies very close to the carrier frequency ($\omega \approx \omega_0$), and propagation constants very close to the propagation constant of the carrier wave $\beta \approx \beta_0 = \beta(\omega_0)$:

$$\vec{E}(\vec{r}, t) = \mathrm{Re}\left(E_0 F(z, t) A(x, y) \exp(i\omega_0 t - i\beta_0 z)\right)\hat{y},$$

Here $\beta(\omega)$ is the spreading constant found in part A, $A(x, y)$ is also taken from part A, $F(z, t)$ is a complex function that contains information about the momentum profile.

The function $\beta(\omega)$ can be approximated by a quadratic function:

$$\beta = \beta_0 + \beta_1(\omega - \omega_0) + \beta_2(\omega - \omega_0)^2/2$$



We can substitute this type of solution into the wave equation, which was derived in the introduction, and by mathematical transformations beyond the school program, obtain the equation for $F(z,t)$;

$$\frac{\partial F}{\partial z}+\beta_1\frac{\partial F}{\partial t}+\frac{i\beta_2}{2}\frac{\partial^2 F}{\partial t^2}-i\gamma|F|^2F=0$$

The pulse will propagate with group velocity, so it is reasonable to move to a frame of reference that will also move with group velocity. In this system the last equation will be rewritten in the form:

$$\frac{\partial F}{\partial z}+\frac{i\beta_2}{2\beta_1^2}\frac{\partial^2 F}{\partial s^2}=i\gamma|F|^2F,$$

where $s=t/\beta_1-z$ – is the spatial coordinate in the soliton reference frame, and γ is the cubic nonlinearity coefficient, which is proportional to χ (и имеет тот же знак). The resulting equation is called the **Nonlinear Schrödinger equation** (hereafter referred to as the NLSE). In the Part D assignment, you will **analyze the NLSE**. You will need to express all answers through its coefficients.

D1	Express the group velocity v_g through β_1.	**0.1**

D2	Let us find the form of the soliton. The solution of NLSE can be found in the form: $$F(z,s)=\frac{F_0\exp(i\sigma z)}{\cosh(\theta s)}.$$ Experess F_0 и σ in terms of θ,β_1,β_2 и γ.	**0.6**

Let one of the natural frequencies of FR be ω_0. The length of FR is such that there are many other natural frequencies in the range from 0 to ω_0 находится очень много других собственных частот. Then the following equality is true for the other natural frequencies: $\omega_\mu\approx\omega_0+D_1\mu+D_2\mu^2/2$, where μ is integer and $\omega_0\gg D_1\gg D_2$.

D3	Express D_1 and D_2 in terms of β_1,β_2 and loop length L. *Hint: the BP natural frequency criterion:* $B_{in}(t)$ *and* $B_{in}(t-\tau(\omega_\mu))$ *have the same phase.*	**0.5**

D4	Let the resonator be made of a material with $\chi>0$. At what D_2 can solitons exist in it?	**0.3**

D5	Let a soliton with carrier frequency ω_0. circulates in the FR described in D3. The external laser does not work. Plot the emission spectrum of the resonator (the dependence of specific power on frequency) **qualitatively** in the frequency range $(\omega_0-20D_1,\omega_0+20D_1)$. Consider that $\omega_0/Q(\omega_0)\ll D_1$. (Remember that Q -is sharpness defined in B7)	**0.6**

D6	Estimate the absolute error of the angular frequency measurement ω using the spectrum from item D5.	**0.2**

D7	Express the round-trip time τ_s through D_1.	**0.2**

Reference materials

Let us introduce a Cartesian coordinate system $Oxyz$. in space. The unit vectors along the corresponding axes are $\widehat{x}, \widehat{y}, \widehat{z}$ and form the right triple.

- The grad operator converts a scalar φ into a vector as follows:

$$\mathrm{grad}\,\varphi = \widehat{x}\frac{\partial\varphi}{\partial x} + \widehat{y}\frac{\partial\varphi}{\partial y} + \widehat{z}\frac{\partial\varphi}{\partial z}$$

- The div operator converts vector $\vec{a} = a_x\widehat{x} + a_y\widehat{y} + a_z\widehat{z}$ into a scalar as follows:

$$\mathrm{div}\,\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

- The rot operator converts vector $\vec{a} = a_x\widehat{x} + a_y\widehat{y} + a_z\widehat{z}$ into a vector as follows:

$$\mathrm{rot}\,\vec{a} = \widehat{x}\Big(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\Big) + \widehat{y}\Big(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\Big) + \widehat{z}\Big(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\Big)$$

- The Laplace operator Δ converts a scalar to a scalar or a vector to a vector:

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

$$\Delta\vec{a} = \frac{\partial^2\vec{a}}{\partial x^2} + \frac{\partial^2\vec{a}}{\partial y^2} + \frac{\partial^2\vec{a}}{\partial z^2}$$

- If we apply rot twice to the vector $\vec{a}$ we get the vector:

$$\mathrm{rot}\,\mathrm{rot}\,\vec{a} = \mathrm{grad}\,\mathrm{div}\,\vec{a} - \Delta\vec{a}.$$

Maxwell's equations

Each equation is written first in integral form and then in differential form:

- Gauss's theorem for magnetic field induction (the flux of magnetic induction through any closed surface is 0):

$$\oint_S (\vec{B}\cdot d\vec{S}) = 0$$

$$\mathrm{div}\,\vec{B} = 0$$

- Gauss' theorem for electric induction (the flux of electric induction through a closed surface is equal to the free charge inside the volume bounded by it):

$$\oint_S (\vec{D}\cdot d\vec{S}) = q_f = \oint_V \rho_f dV$$

$$\mathrm{div}\,\vec{D} = \rho_f$$

here ρ_f is the bulk density of free charges.

- Faraday's law of electromagnetic induction (the rate of change of the flux of magnetic induction through an unclosed surface, taken with the opposite sign, is equal to the circulation of the electric field on a closed loop, which is the boundary of the surface):

$$\oint_l (\vec{E}\cdot d\vec{l}) = -\frac{\partial}{\partial t}\oint_S (\vec{B}\cdot d\vec{S})$$

$$\mathrm{rot}\,\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

- Circulation theorem for the magnetic field strength (the sum of the rate of change of the electric induction flux through an unclosed surface and the total electric current of free charges through it is equal to the magnetic field circulation on a closed contour, which is the boundary of the surface):

$$\oint_l (\vec{H} \cdot d\vec{l}) = I_f + \frac{\partial}{\partial t} \oint_S (\vec{D} \cdot d\vec{S})$$

$$\operatorname{rot} \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j}_f$$